

Super compact equation for water waves

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We derive very simple compact equation for gravity water waves which includes nonlinear wave term (à la NLSE) and advection term (may results in wave breaking).

1. Introduction

A potential flow of an ideal incompressible fluid with free surface in a gravity field is described (Zakharov 1968) by the following Hamiltonian system:

$$\frac{\partial \psi}{\partial t} = -\frac{\delta H}{\delta \eta} \quad \frac{\partial \eta}{\partial t} = \frac{\delta H}{\delta \psi}. \quad (1.1)$$

Thereafter we study only the case of one horizontal direction. Now

$$\begin{aligned} \eta &= \eta(x, t) - \text{shape of the surface,} \\ \psi &= \psi(x, t) = \phi(x, \eta(x, t), t) - \text{potential on the surface,} \\ \phi(x, z, t) &- \text{potential inside the fluid.} \end{aligned} \quad (1.2)$$

The Hamiltonian H is

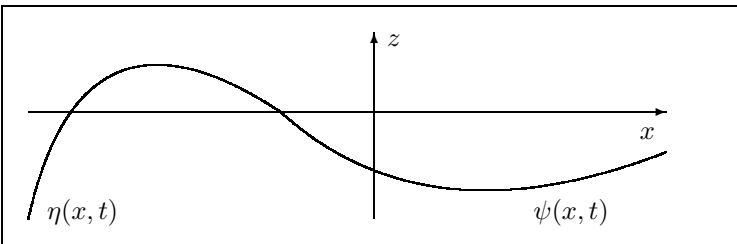
$$H = \frac{1}{2} \int dx \int_{-\infty}^{\eta} |\nabla \phi|^2 dz + \frac{g}{2} \int \eta^2 dx \quad (1.3)$$

The potential $\phi(x, z, t)$ satisfies the Laplace equation:

$$\frac{\partial^2 \phi}{\partial^2 x} + \frac{\partial^2 \phi}{\partial^2 z} = 0$$

with the asymptotic boundary conditions:

$$\frac{\partial \phi}{\partial z} \rightarrow 0, \quad \text{at } z \rightarrow -\infty.$$



If the steepness of surface is small, $\eta_x^2 \ll 1$, the Hamiltonian can be presented by the

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infinite series

$$\begin{aligned}
H &= H_2 + H_3 + H_4 + \dots \\
H_2 &= \frac{1}{2} \int (g\eta^2 + \psi \hat{k} \psi) dx, \\
H_3 &= -\frac{1}{2} \int \{(\hat{k}\psi)^2 - (\psi_x)^2\} \eta dx, \\
H_4 &= \frac{1}{2} \int \{\psi_{xx} \eta^2 \hat{k} \psi + \psi \hat{k} (\eta \hat{k} (\eta \hat{k} \psi))\} dx.
\end{aligned} \tag{1.4}$$

where $\hat{k}\psi$ means multiplication by $|k|$ in k -space ($|k| = \sqrt{-\frac{\partial^2}{\partial x^2}}$).

Equations (1.1) although truncated according to (1.4) even for the full 3-D case can be efficiently used for numerical simulations of water wave dynamics (see, for instance (Korotkevich et al 2008)). However, they are not convenient for analytic study because $\eta(x, t)$ and $\psi(x, t)$ are not "optimal" canonical variables. One can choose better Hamiltonian variables by performing a proper canonical transformation. This transformation can be done in two steps. In the first step we eliminate cubic terms in the Hamiltonian and simplify essentially quartic terms. What we obtain as a result of this transformation is so called "Zakharov equation which was widely used in recent years by many researchers (see, for instance (Crawford et al 1980; Debnath 1994)) of more recent publications (Annenkov & Shrira 2011, 2013). In the second step one can "improve" Zakharov equation applying appropriate canonical transformation. This "improvement" is possible due to some very special property of the quartic Hamiltonian in Zakharov equation. We mean mysterious cancellation (Dyachenko & Zakharov 1994) of nontrivial four-wave interactions. This cancellation takes place only in one-dimensional case. this cancellation makes possible to replace the "generic" Zakharov equation by much more suitable "compact equation", (Dyachenko & Zakharov 2011, 2012), which was intensively used as a base for both numerical simulations (Fedele & Dutykh 2012,a; Dyachenko 2013; Dyachenko et al 2014; Fedele 2014a,b; Dyachenko et al 2016, 2015a,b) and analytical proof on nonintegrability Of Zakharov equation (Dyachenko et al 2013a).

In this paper we discovered that the second step in the canonical transformation is not a unique procedure. One can do it by many different ways, obtaining different forms of the compact equation. In this paper we present the most optimal (by our opinion) version of the compact equation which we call "the super compact equation" for water waves. We present also some preliminary results of numerical simulations made by the use of this equation.

2. Zakharov equation

Here we briefly recall how to obtain Zakharov equation starting with Hamiltonian (1.4). All the detail can be found in Zakharov (1968); Krasitskii (1990); Zakharov et al (1992).

So, Zakharov equation can be derived in two steps.

1. It is convenient to introduce normal complex variable a_k :

$$\eta_k = \sqrt{\frac{\omega_k}{2g}} (a_k + a_{-k}^*) \quad \psi_k = -i \sqrt{\frac{g}{2\omega_k}} (a_k - a_{-k}^*) \tag{2.1}$$

here $\omega_k = \sqrt{gk}$ -is the dispersion law for the gravity waves, and Fourier transformations

$\psi(x) \rightarrow \psi_k$ and $\eta(x) \rightarrow \eta_k$ are defined as follows:

$$f_k = \frac{1}{\sqrt{2\pi}} \int f(x) e^{-ikx} dx, \quad f(x) = \frac{1}{\sqrt{2\pi}} \int f_k e^{+ikx} dk.$$

With a_k the Hamiltonian takes the form:

$$\begin{aligned} H_2 &= \int \omega_k a_k a_k^* dk, \\ H_3 &= \int V_{k_1 k_2}^k \{a_k^* a_{k_1} a_{k_2} + a_k a_{k_1}^* a_{k_2}^*\} \delta_{k-k_1-k_2} dk dk_1 dk_2 \\ &\quad + \frac{1}{3} \int U_{k k_1 k_2} \{a_k a_{k_1} a_{k_2} + a_k^* a_{k_1}^* a_{k_2}^*\} \delta_{k+k_1+k_2} dk dk_1 dk_2, \\ H_4 &= \frac{1}{2} \int W_{k_1 k_2}^{k_3 k_4} a_{k_1}^* a_{k_2}^* a_{k_3} a_{k_4} \delta_{k_1+k_2-k_3-k_4} dk_1 dk_2 dk_3 dk_4 + \\ &\quad + \frac{1}{3} \int G_{k_1 k_2 k_3}^{k_4} (a_{k_1}^* a_{k_2}^* a_{k_3}^* a_{k_4} + c.c.) \delta_{k_1+k_2+k_3-k_4} dk_1 dk_2 dk_3 dk_4 + \\ &\quad + \frac{1}{12} \int R_{k_1 k_2 k_3 k_4} (a_{k_1}^* a_{k_2}^* a_{k_3}^* a_{k_4}^* + c.c.) \delta_{k_1+k_2+k_3+k_4} dk_1 dk_2 dk_3 dk_4 \end{aligned}$$

Explicit expressions for coefficients of Hamiltonian are not important here. Nevertheless they can be found in Zakharov (1998, 1999); Dyachenko et al (2016). The motion equations (1.1) now take the form:

$$\frac{\partial a_k}{\partial t} + \frac{\delta H}{\delta a_k^*} = 0. \quad (2.2)$$

2. Variables a_k are still not optimal. For transition to better variables b_k one has to perform a canonical transformation $a_k \rightarrow b_k$ to cancel all nonresonant cubic and quartic terms in the new Hamiltonian. The most economical way to construct the transformation was offered in (Zakharov et al 1992).

By performing the transformation we end up with the Hamiltonian

$$H = \int \omega_k b_k b_k^* dk + \frac{1}{2} \int T_{k k_1}^{k_2 k_3} b_k^* b_{k_1}^* b_{k_2} b_{k_3} \delta_{k+k_1-k_2-k_3} dk dk_1 dk_2 dk_3 + \tilde{H} \quad (2.3)$$

\tilde{H} is an infinite series in b_k, b_k^* starting from the fifth order terms. The explicit (and cumbersome) expression for $T_{k k_1}^{k_2 k_3}$ can be found in (Zakharov 1968; Zakharov 1998, 1999). The motion equation

$$\frac{\partial b_k}{\partial t} + \frac{\delta H}{\delta b_k^*} = 0. \quad (2.4)$$

(neglecting \tilde{H}) is the traditional Zakharov equation.

3. Canonical transformation for Zakharov equation

A possibility of further simplification of equation (2.4) is based on the remarkable fact, established in (Dyachenko & Zakharov 1994). It is the following. Let us consider the resonant condition for four wave interaction

$$\begin{aligned} k + k_1 &= k_2 + k_3, \\ \omega_k + \omega_{k_1} &= \omega_{k_2} + \omega_{k_3}, \end{aligned} \quad (3.1)$$

In 1-D case this system of equations can be resolved as follows:

$$k = a(1 + \zeta)^2,$$

$$\begin{aligned} k_1 &= a(1 + \zeta)^2 \zeta^2, \\ k_2 &= -a\zeta^2, \\ k_3 &= a(1 + \zeta + \zeta^2)^2 \quad \text{here } 0 < \zeta < 1. \end{aligned} \quad (3.2)$$

Notice that $kk_1k_2k_3, 0$. Now

$$T_{kk_1}^{k_2k_3} = F(a, \zeta) = a^3 f(\zeta).$$

Direct calculation shows that

$$f(\zeta) \equiv 0. \quad (3.3)$$

This fact means that "nontrivial" four-wave resonances are absent. However system (3.1) has also "trivial" solution:

$$k_2 = k_1, \quad k_3 = k, \quad \text{or} \quad k_2 = k, \quad k_3 = k_1, \quad (3.4)$$

We introduce T_{kk_1} (diagonal part) as value of the four-wave coefficient on the trivial manifold (3.4). It was calculated in (Zakharov 1968) and is equal to:

$$T_{kk_1} = T_{kk_1}^{kk_1} = \frac{1}{4\pi} |k| |k_1| (|k + k_1| - |k - k_1|) = \frac{1}{2\pi} |k| |k_1| \min(|k|, |k_1|)$$

Let us introduce $\tilde{T}_{kk_1}^{k_2k_3}$ as follows:

$$\tilde{T}_{kk_1}^{k_2k_3} = \theta(kk_1k_2k_3) \left[\frac{1}{2} (T_{kk_2} + T_{kk_2} + T_{k_1k_2} + T_{k_1k_3}) - \frac{1}{4} (T_{kk} + T_{k_1k_1} + T_{k_2k_2} + T_{k_3k_3}) \right] \quad (3.5)$$

Here $\theta(k)$ is the step-function. Canonical transformation of the second step has to replace cumbersome Zakharov's $T_{kk_1}^{kk_1}$ from (2.3) by much more simple $\tilde{T}_{kk_1}^{kk_1}$. Obviously their diagonal parts are the same.

The simple method to construct canonical transformation is based on the fact that a Hamiltonian system keeps at all times Hamiltonian properties. It means that transformation $c_k(0) \rightarrow c_k(\tau)$ is canonical. Let us construct this transformation (as a power series) using some auxiliary Hamiltonian \tilde{H} (starting from the quartic term) of the form:

$$\tilde{H} = \frac{1}{2} \int \tilde{\mathbf{B}}_{\mathbf{kk}_1}^{\mathbf{k}_2\mathbf{k}_3} c_k^* c_{k_1}^* c_{k_2} c_{k_3} \delta_{k+k_1-k_2-k_3} dk dk_1 dk_2 dk_3 + \dots \quad (3.6)$$

Obviously

$$\tilde{\mathbf{B}}_{\mathbf{kk}_1}^{\mathbf{k}_2\mathbf{k}_3} = \tilde{\mathbf{B}}_{\mathbf{k}_1\mathbf{k}}^{\mathbf{k}_2\mathbf{k}_3} = \tilde{\mathbf{B}}_{\mathbf{kk}_1}^{\mathbf{k}_3\mathbf{k}_2} = (\tilde{\mathbf{B}}_{\mathbf{k}_2\mathbf{k}_3}^{\mathbf{kk}_1})^*$$

Using Taylor series we can express old canonical $b_k(\tau) = c_k(\tau)$ in terms of $c_k(0)$:

$$\begin{aligned} c_k(\tau) &= c_k(0) + \tau \dot{c}_k(0) + \dots \\ \dot{c}_k(0) &= -i \frac{\delta \tilde{H}(c_k(0), c_k^*(0))}{\delta c_k^*(0)} \end{aligned}$$

and

$$b_k = c_k - i \int \tilde{\mathbf{B}}_{\mathbf{kk}_1}^{\mathbf{k}_2\mathbf{k}_3} c_{k_1}^* c_{k_2} c_{k_3} \delta_{k+k_1-k_2-k_3} dk_1 dk_2 dk_3 + \dots$$

Now we plug this transformation in the Hamiltonian (2.3) of Zakharov equation and get new Hamiltonian:

$$\begin{aligned} H &= \int \omega_k c_k c_k^* dk + \frac{1}{2} \int \left[T_{kk_1}^{k_2k_3} - i(\omega_k + \omega_{k_1} - \omega_{k_2} - \omega_{k_3}) \tilde{\mathbf{B}}_{\mathbf{kk}_1}^{\mathbf{k}_2\mathbf{k}_3} \right] \times \\ &\quad \times c_k^* c_{k_1}^* c_{k_2} c_{k_3} \delta_{k+k_1-k_2-k_3} dk dk_1 dk_2 dk_3 + \dots \end{aligned} \quad (3.7)$$

Coefficient $\tilde{\mathbf{B}}_{\mathbf{k}\mathbf{k}_1}^{\mathbf{k}_2\mathbf{k}_3}$ of the auxiliary Hamiltonian is also the coefficient of canonical transformation. It controls the four-wave coefficient $T_{kk_1}^{k_2k_3}$ in the Hamiltonian of Zakharov equation (3.7). To replace cumbersome $T_{kk_1}^{k_2k_3}$ by more simple $\tilde{T}_{kk_1}^{k_2k_3}$, $\tilde{\mathbf{B}}_{\mathbf{k}\mathbf{k}_1}^{\mathbf{k}_2\mathbf{k}_3}$ has to be equal to:

$$\tilde{\mathbf{B}}_{\mathbf{k}\mathbf{k}_1}^{\mathbf{k}_2\mathbf{k}_3} = i \frac{\tilde{T}_{kk_1}^{k_2k_3} - T_{kk_1}^{k_2k_3}}{\omega_k + \omega_{k_1} - \omega_{k_2} - \omega_{k_3}}. \quad (3.8)$$

One can check that $\tilde{\mathbf{B}}_{\mathbf{k}\mathbf{k}_1}^{\mathbf{k}_2\mathbf{k}_3}$ has no singularities at $k + k_1 = k_2 + k_3$. Indeed in the area where $kk_1k_2k_3 \leq 0$ singularities are canceled in virtue of identity (3.3). In the area where $kk_1k_2k_3 > 0$ singularities are canceled due to special choice of $\tilde{T}_{kk_1}^{k_2k_3}$. Explicit expression for $\tilde{\mathbf{B}}_{\mathbf{k}\mathbf{k}_1}^{\mathbf{k}_2\mathbf{k}_3}$ was published in (Dyachenko et al 1995). By this way we derive the "compact water wave equation".

Due to the absence of nontrivial resonances, waves moving in the same direction do not generate waves moving in the opposite direction. Hence we can assume that all $k_i > 0$. Finally

$$\begin{aligned} \tilde{T}_{kk_1}^{k_2k_3} = & \left[-\frac{1}{8\pi}(kk_2|k-k_2| + kk_3|k-k_3| + k_1k_2|k_1-k_2| + k_1k_3|k_1-k_3|) + \right. \\ & \left. + \frac{1}{8\pi}(kk_1(k+k_1) + k_2k_3(k_2+k_3)) \right] \theta(k)\theta(k_1)\theta(k_2)\theta(k_3) \end{aligned} \quad (3.9)$$

It corresponds to the following Hamiltonian in x -space:

$$H = \int b^* \hat{\omega}_k b dx + \frac{1}{2} \int \left| \frac{\partial b}{\partial x} \right|^2 \left[\frac{i}{2} \left(b \frac{\partial b^*}{\partial x} - b^* \frac{\partial b}{\partial x} \right) - \hat{k} |b|^2 \right] dx. \quad (3.10)$$

Here we again went back to using variable $b(x, t)$. The compact equation with the Hamiltonian (3.10) was used as a base for numerical Simulations in papers ()

4. Super compact equation

Now we notice that choice (3.5) is not a unique way for introducing a new Hamiltonian. In fact, the conditions imposed on $\tilde{T}_{kk_1}^{k_2k_3}$ are pretty loose. They are

(i) Symmetry conditions. One must demand that

$$\tilde{\mathbf{T}}_{\mathbf{k}\mathbf{k}_1}^{\mathbf{k}_2\mathbf{k}_3} = \tilde{\mathbf{T}}_{\mathbf{k}_1\mathbf{k}}^{\mathbf{k}_2\mathbf{k}_3} = \tilde{\mathbf{T}}_{\mathbf{k}\mathbf{k}_1}^{\mathbf{k}_3\mathbf{k}_2} = \tilde{\mathbf{T}}_{\mathbf{k}_2\mathbf{k}_3}^{\mathbf{k}\mathbf{k}_1}.$$

(ii) The diagonal part must be strictly defined

$$\tilde{\mathbf{T}}_{\mathbf{k}\mathbf{k}_1}^{\mathbf{k}_2\mathbf{k}_3} = T_{kk_1} = \frac{1}{4\pi} |k| |k_1| (|k+k_1| - |k-k_1|).$$

Let us choose $\tilde{\mathbf{T}}_{\mathbf{k}\mathbf{k}_1}^{\mathbf{k}_2\mathbf{k}_3}$ as follows:

$$\begin{aligned} \tilde{T}_{k_2k_3}^{kk_1} &= \frac{(kk_1k_2k_3)^{\frac{1}{2}}}{2\pi} \min(k, k_1, k_2, k_3) \times \theta_k \theta_{k_1} \theta_{k_2} \theta_{k_3} \\ \min(k, k_1, k_2, k_3) &= \frac{1}{4} (k+k_1+k_2+k_3 - |k-k_2| - |k-k_3| - |k_1-k_2| - |k_1-k_3|) \\ \text{here } \theta_k &- \text{ is the step-function, } \theta_k = \theta(k) \end{aligned} \quad (4.1)$$

Now function b_k satisfies the equation:

$$i\dot{b}_k = \frac{\delta H}{\delta b_k^*} = \omega_k b_k + \frac{k^{\frac{1}{2}} \theta_k}{2\pi} \int \min(k, k_1, k_2, k_3) c_{k_1}^* c_{k_2} c_{k_3} \delta_{k+k_1-k_2-k_3} dk_1 dk_2 dk_3 \quad (4.2)$$

where

$$c_k = k^{\frac{1}{2}} \theta_k b_k$$

is Fourier-image of analytical (in the upper half-plane) function. Note, nonlinear term in (4.2) preserve this property. Multiplying (4.2) by $ik^{\frac{1}{2}}$ one can easily get:

$$\dot{c}_k + ik\theta_k \left[\frac{\omega_k}{k} c_k + \frac{1}{2\pi} \int \min(k, k_1, k_2, k_3) c_{k_1}^* c_{k_2} c_{k_3} \delta_{k+k_1-k_2-k_3} dk_1 dk_2 dk_3 \right] = 0 \quad (4.3)$$

Expression in square brackets of (4.3) is variational derivative of the following Hamiltonian:

$$H = \int \frac{\omega_k}{k} |c_k|^2 dk + \frac{1}{4\pi} \int \min(k, k_1, k_2, k_3) c_k^* c_{k_1}^* c_{k_2} c_{k_3} \delta_{k+k_1-k_2-k_3} dk dk_1 dk_2 dk_3 \quad (4.4)$$

Using following relations between k -space and x -space

$$\begin{aligned} kc_k^* &\Leftrightarrow i \frac{\partial}{\partial x} c^*(x), & kc_k &\Leftrightarrow -i \frac{\partial}{\partial x} c(x), \\ |k - k_2| c_k^* c_{k_2} &\Leftrightarrow \hat{K}(|c(x)|^2), & (k + k_1) c_k c_{k_1} &\Leftrightarrow -i \frac{\partial}{\partial x} (c(x)^2), \end{aligned}$$

Relation (4.3) is exactly our super compact equation.

Hamiltonian can be written in x -space:

$$H = \int c^* \hat{V} c dx + \frac{1}{2} \int \left[\frac{i}{4} (c^2 \frac{\partial}{\partial x} c^{*2} - c^{*2} \frac{\partial}{\partial x} c^2) - |c|^2 \hat{K}(|c|^2) \right] dx \quad (4.5)$$

Here operator \hat{V} in K -space is so that $V_k = \frac{\omega_k}{k}$. If along with this to introduce bracket similar to Gardner-Zakharov-Faddeev

$$\partial_x^+ \Leftrightarrow ik\theta_k \quad (4.6)$$

than equation of motion is the following:

$$\frac{\partial c}{\partial t} + \partial_x^+ \frac{\delta H}{\delta c^*} = 0. \quad (4.7)$$

Introducing advection velocity

$$\mathcal{U} = \hat{K}|c|^2 \quad (4.8)$$

taking variational derivative one can write the equation (4.7) in the form:

$$\frac{\partial c}{\partial t} + i\hat{\omega}c - i\partial_x^+ \left(|c|^2 \frac{\partial c}{\partial x} \right) = \partial_x^+ (\mathcal{U}c) \quad (4.9)$$

one can recognize two terms in the equation:

- nonlinear waving: $i\hat{\omega}c - i\partial_x^+ (|c|^2 \frac{\partial c}{\partial x})$
- advection term: $\partial_x^+ (\mathcal{U}c)$.

Along with usual quantities such as energy and both momenta equation (4.9) conserves action or number of waves:

$$N = \int \frac{|c|^2}{k} dx.$$

Equation (4.9) has exact self-similar substitution

$$c(x, t) = g(t_0 - t)^{\frac{3}{2}} C \left(\frac{x}{g(t_0 - t)^2} \right).$$

Easy to check that $C(\xi)$ satisfies the following equation:

$$\frac{3}{2}C - 2\xi \frac{\partial C}{\partial \xi} + i\hat{K}^{\frac{1}{2}}C - i \frac{\partial}{\partial \xi} \left(|C|^2 \frac{\partial C}{\partial \xi} \right) = \frac{\partial}{\partial \xi} \left((\hat{K}|C|^2)C \right) \quad (4.10)$$

where $C(\xi)$ - is dimensionless function which is analytic in the upper half-plane, \hat{K} - is dimensionless operator.

In k -space equation (4.3) has the following solution:

$$c(k, t) = g^2(t_0 - t)^{\frac{7}{2}} F(gk(t_0 - t)^2) \quad (4.11)$$

Easy to check that dimensionless function $F(\xi)$ satisfies the following equation:

$$\frac{7}{2}F + 2\xi \frac{\partial F}{\partial \xi} = i\xi^{\frac{1}{2}}F + \frac{i\xi}{2\pi} \int \min(\xi, \xi_1, \xi_2, \xi_3) F^*(\xi_1) F(\xi_2) F(\xi_3) \delta_{\xi+\xi_1-\xi_2-\xi_3} d\xi_1 d\xi_2 d\xi_3 \quad (4.12)$$

5. Back to η and ψ

According to canonical transformation η_k and ψ_k are power series of b_k (or c_k) up to the third order:

$$\eta_k = \eta_k^{(1)} + \eta_k^{(2)} + \eta_k^{(3)}, \quad \psi_k = \psi_k^{(1)} + \psi_k^{(2)} + \psi_k^{(3)}. \quad (5.1)$$

Details of the recovering physical quantities $\eta(x, t)$ and $\psi(x, t)$ are given in Dyachenko et al (2016). Obviously

$$\eta_k^{(1)} = \sqrt{\frac{\omega_k}{2g}} [b_k + b_{-k}^*], \quad \psi_k^{(1)} = -i \sqrt{\frac{g}{2\omega_k}} [b_k - b_{-k}^*]. \quad (5.2)$$

Or

$$\eta^{(1)}(x) = \frac{1}{\sqrt{2g^{\frac{1}{4}}}} (\hat{k}^{\frac{1}{4}} b(x) + \hat{k}^{\frac{1}{4}} b(x)^*), \quad \psi^{(1)}(x) = -i \frac{g^{\frac{1}{4}}}{\sqrt{2}} (\hat{k}^{-\frac{1}{4}} b(x) - \hat{k}^{-\frac{1}{4}} b(x)^*). \quad (5.3)$$

Operators \hat{k}^α act in Fourier space as multiplication by $|k|^\alpha$.

$$\begin{aligned} \eta^{(2)}(x) &= \frac{\hat{k}}{4\sqrt{g}} [\hat{k}^{\frac{1}{4}} b(x) - \hat{k}^{\frac{1}{4}} b^*(x)]^2, \\ \psi^{(2)}(x) &= \frac{i}{2} [\hat{k}^{\frac{1}{4}} b^*(x) \hat{k}^{\frac{3}{4}} b^*(x) - \hat{k}^{\frac{1}{4}} b(x) \hat{k}^{\frac{3}{4}} b(x)] + \\ &\quad + \frac{1}{2} \hat{H} [\hat{k}^{\frac{1}{4}} b(x) \hat{k}^{\frac{3}{4}} b^*(x) + \hat{k}^{\frac{1}{4}} b^*(x) \hat{k}^{\frac{3}{4}} b(x)]. \end{aligned} \quad (5.4)$$

Here \hat{H} - is Hilbert transformation with eigenvalue $i \text{sign}(k)$.

6. Numerical Simulation

6.1. Breather

Breather is the localized solution of the following type:

$$c(x, t) = C(x - Vt) e^{i(k_0 x - \omega_0 t)} \quad \text{or} \quad c_k = e^{i(\Omega + Vk)t} \phi_k$$

where ϕ_k satisfies the equation:

$$(\Omega + Vk - \omega_k) \phi_k = \frac{1}{2} \int T_{k k_1}^{k_2 k_3} \phi_{k_1}^* \phi_{k_2} \phi_{k_3} \delta_{k+k_1-k_2-k_3} dk_1 dk_2 dk_3$$

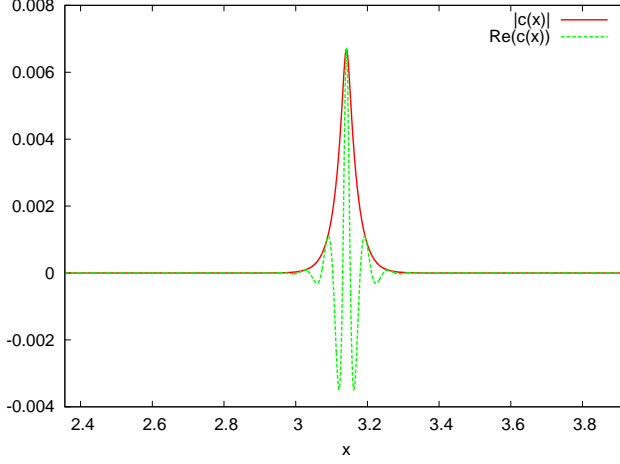
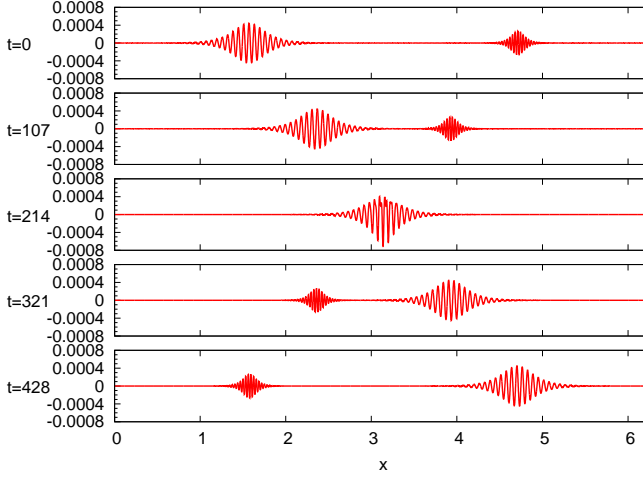
FIGURE 1. Narrow breather with three crests. $\text{Re}(c(x,0))$ and $|c(x,0)|$ 

FIGURE 2. Snapshots of breather collision

It can be found by Petviashvili method

$$\phi_k^{n+1} = \frac{NL_k^n}{M_k} \left[\frac{\langle \phi^n \cdot NL(\phi^n) \rangle}{\phi^n \cdot M \phi^n} \right]^\gamma, \quad M_k = \Omega + V k - \omega_k,$$

$$NL(\phi^n) = -P^+ \frac{\partial}{\partial x} \left(|\phi^n|^2 \frac{\partial \phi^n}{\partial x} \right) + i P^+ \frac{\partial}{\partial x} \left(\hat{k} (|\phi^n|^2) \phi^n \right)$$

Breather solution of this equation in the periodic domain 2π with $k_0 = 100$ is shown in Fig.1. Breather is very stable structure. Collision of two breathers moving with different velocities (or with $k_0 = 100$ and $k_0 = 200$) is shown in Fig.2.

6.2. Modulational instability

Freak-wave appearing from homogeneous sea with $k_0 = 100$ and steepness $\mu = 0.085$ in the Fig.3:

One can see the beginning of wave breaking in the Fig.4:

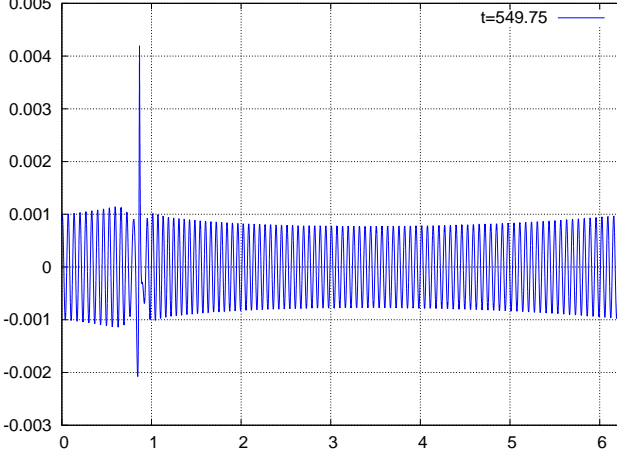


FIGURE 3. Freak-wave

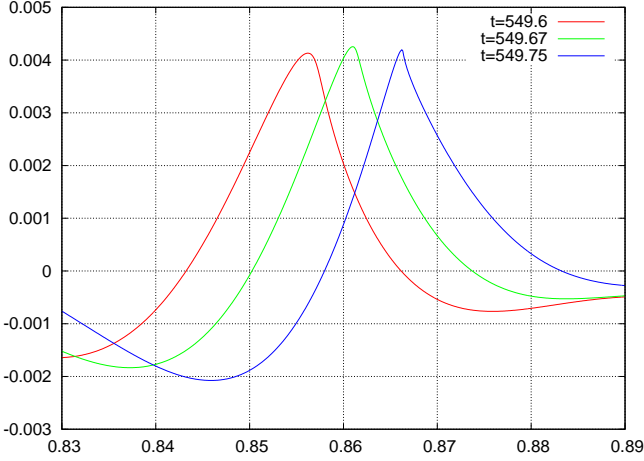


FIGURE 4. Three snapshots showing beginning of wave breaking

7. Conclusion

We derive new compact and elegant form of Hamiltonian and equation for the gravity waves at the surface of deep water.

- Equation is written for complex normal variable $c(x, t)$ which is analytic function in the upper half-plane
- Hamiltonian both in k -space (4.4) and in x -space (4.5) is very simple
- Equation itself is very straightforward consisting of only two terms - nonlinear waves and advection
- It can be easily implemented for numerical simulation

The equation can be generalized for "almost" 2-D waves like KdV is generalized to Kadomtsev-Petviashvili equation:

$$H = \int c^* \hat{V} c \, dx dy + \frac{1}{2} \int \left[\frac{i}{4} \left(c^2 \frac{\partial}{\partial x} c^{*2} - c^{*2} \frac{\partial}{\partial x} c^2 \right) - |c|^2 \hat{K}_x (|c|^2) \right] dx dy \quad (7.1)$$

Here operator \hat{V} in K-space is $V_{\vec{k}} = \frac{\omega_{\vec{k}}}{k_x}$.

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